The Open University of Sri Lanka
B.Sc./B.Ed. Degree Programme
Final Examination-2012/2013
APU 3143/APE5143-Mathematical Methods
Applied Mathematics -Level 05

Duration: Two Hours.

Date: 26.06.2013	**Time: 1.00 p.m.- 3.00p.m.**

Answer FOUR questions only.

1. (i) The Laplace transform of a function $f(t)$, denoted by $L[f(t)]$ is defined as

$$L[f(t)] = F(s) = \int_0^\infty f(t)e^{-st}dt.$$

Show the following using the definition of L:

(a) $L^{-1}\left\{\dfrac{e^{-3s}}{s^2-2s+5}\right\} = \dfrac{1}{2}e^{(t-3)}\sin 2(t-3).u(t-3),$

(b) $L^{-1}\left\{\dfrac{e^{-3s}}{(s-4)^2}\right\} = (t-3)e^{4(t-3)}u(t-3),$

where $u(t)$ has the standard meaning.

(ii) Solve each of the following boundary value problems using the Laplace transform method.

(a) $\dfrac{d^2y}{dt^2} + y = e^{-2t}\sin t,$ subject to $y(0) = 0,\ y'(0) = 0.$

(b) $\dfrac{d^3y}{dt^3} + 4\dfrac{d^2y}{dt^2} + 5\dfrac{dy}{dt} + 2y = 10\cos t,$ subject to $y(0) = 0,\ y'(0) = 0,\ y''(0) = 3.$

2. Obtain the formal expansion of the function f defined by $f(x) = \ln x,\ \ 1 \le x \le e^{2\pi}$ as a series of orthonormal characteristic functions $\{\phi_n\}$ of the Sturm-Liouville problem

$$\frac{d}{dx}\left[x\frac{dy}{dx}\right] + \frac{\lambda}{x}y = 0$$

$$y(1) = 0$$

$$y(e^{\pi}) = 0$$

where $\lambda \ge 0$.

3. For each of the following functions $f(x)$, find the Fourier Sine series and the Fourier Cosine series in $0 \le x \le L$:

(i) $f(x) = 4x \quad ;0 \le x \le L$

(ii) $f(x) = \begin{cases} \dfrac{x}{10} & ;0 \le x \le L/2 \\ \dfrac{L-x}{10} & ;L/2 \le x \le L \end{cases}$

4. (i) The Gamma function denoted by $\Gamma(p)$ corresponding to the parameter p is defined by the improper integral $\Gamma(p) = \int_0^{\infty} e^{-t} t^{p-1} dt, \quad (p > 0).$

Show the following:

(a) $\int_0^{\infty} x^{-\frac{3}{2}}\left(1 - e^{-x}\right) dx = 2\sqrt{\pi}.$ (You may assume that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.)

(b) $\int_0^{\infty} \frac{x^a}{a^x} dx = \frac{\Gamma(a+1)}{(\ln a)^{a+1}}; \quad a > 0.$

(ii) The Beta function denoted by $\beta(p,q)$ is defined by $\beta(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1} dx,$ where $p>0$ and $q>0$ are parameters.

Prove the following:

(a) $\int_0^{\frac{\pi}{2}} \sqrt{\cot\theta}\, d\theta = \frac{1}{2}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \frac{1}{2}\pi\sqrt{2}.$

(b) $\int_0^1 x^m (Inx)^n\, dx = \frac{(-1)^n n!}{(m+1)^{n+1}}$ where n is a positive integer and m>-1.

5. Let $J_p(x)$ be the Bessel function of order p given by the expansion

$$J_p(x) = x^p \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+p}.m!\Gamma(p+m+1)}.$$

(i) Evaluate the following integrals:

(a) $\int x^3 J_0(x)\, dx.$ (b) $\int J_3(x)\, dx.$ (c) $\int J_5(x)\, dx.$

(ii) Express $J_{\frac{7}{2}}(x)$ in terms of suitable sine and cosine functions.

Note: You may use related Recurrence Relations for Bessel's Functions toprove the above results.

6. The Rodrigues' formula for the n^{th}degree Legendre polynomial denoted by$P_n(x)$ is given as

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2 - 1)^n.$$

$P_n(x)$ is also given by the sum

$$P_n(x) = \sum_{m=0}^{M} \frac{(-1)^m (2n-2m)!}{2^n m!(n-m)!(n-2m)!}.x^{n-2m}\ , \quad n = 0,\ 1,\ 2,\ldots,$$

where $M = \frac{n}{2}$ or $\frac{n-1}{2}$ whichever is an integer.

Using this expansion prove that

(i) $P_n'(1) = \dfrac{n(n+1)}{2}$

(ii) $P_n'(-1) = (-1)^n \dfrac{n(n+1)}{2}$.

(iii) $\int_{-1}^{1} (1-x^2) P_m' P_n' dx = \begin{cases} 0 & \text{if } m \neq n \\ \dfrac{2n(n+1)}{2n+1} & \text{if } m = n \end{cases}$

Hint: Multiply Legendre's equation for P_m by P_n, integrate -1 to 1 and use orthogonality.