The Open University of Sri Lanka
B.Sc./B.Ed Degree Programme
Pure Mathematics – Level 05
Final Examination-2009/2010
PMU3295/ PME5295- Ring Theory-Paper II

Duration: Two and Half Hours

Date: 15/06/2010 Time: 9.30am-12.00 noon

Answer Four Questions Only.

1. (a) (i) Let I be a proper ideal of the ring R. Then show that I is a prime ideal if and only if the quotient ring R/I is an integral domain.

 (ii) Does the statement in part (i) hold if R is without the identity? Justify your answer.

 (iii) What is the converse of the statement in part (i)? Is the converse of part (i) true? Justify your answer.

 (b) State the correspondence theorem for rings.

2. (a) Show that in an integral domain R, any prime element p is irreducible.

 (b) Show that in a principle ideal domain R, a non –zero element $p \in R$ is irreducible if and only if it is prime.

3. (a) State the definitions of
 (i) nilpotent element
 (ii) idempotent element
 of a ring R.

 (b) Show that the only idempotents of a ring R having exactly one maximal ideal M are 0 and 1.

 (c) If a and b are nilpotent elements of a commutative ring, show that $a+b$ is also nilpotent. Give an example to show that this may fail if the ring is not commutative.

4. Define 'unique factorization domain'.
(a) Let p be an element of a Euclidean ring R, then show that p is irreducible if and only if p is prime.

(b) Let R be a unique factorization domain. Show that, for $a \in R$, a is irreducible if and only if a is prime.

5. (a) Define $N : \mathbb{Z}[i] \to \mathbb{Z}$ by
$$N(\alpha) = \alpha\overline{\alpha}$$
where $\overline{\alpha}$ is the complex conjugate of α. Let $\alpha, \beta \in \mathbb{Z}[i]$. Then show that
(i) $\alpha | \beta$ in $\mathbb{Z}[i]$ implies that $N(\alpha) | N(\beta)$ in $\mathbb{Z}$.
(ii) α is a unit if and only if $N(\alpha) = 1$.
(iii) $N(\alpha)$ is irreducible in $\mathbb{Z}$ implies that α is irreducible in $\mathbb{Z}[i]$.
Is the converse of part (a) true? Justify your answer.

(b) Consider the ring
$$R = \mathbb{Z}\left[\sqrt{-3}\right] = \left\{a + ib\sqrt{3} \,|\, a, b \in \mathbb{Z}\right\}.$$
Prove that 2 is irreducible but not prime.

6. (a) State and prove the Euler-Fermat Theorem.

(b) Show that a commutative ring with identity is a field if and only if it has no non trivial ideals.

(c) Let f be a homomorphism from a field F into F'. Then show that either f is the trivial homomorphism or else f is one-to-one.