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The Open University of Sri Lanka

B.Sc./B.Ed., Continuing Education Degree Programme

Final Examination-2024/2025

ADU5302/ADE5302-Mathematical Methods

Applied Mathematics -Level 05

DURATION: Two (02) Hours

Date: 04.12.2024	**Time:1.30p.m. - 3.30p.m.**

Important: Answer FOUR questions only. Select at least ONE question from part A.

Part A

1. Obtain the formal expansion of the function f defined by $f(x) = x \;\; (0 \le x \le \pi)$ as a series of orthonormal characteristic functions $\{\phi_n\}$ of the Sturm-Liouville problem

$$\frac{d^2y}{dx^2} + \lambda y = 0 \text{ with}$$
$$y(0)=0,$$
$$y(\pi)=0.$$

2. The Rodrigue's formula for the n^{th} degree Legendre polynomial denoted by $P_n(x)$ is given as

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n.$$

$P_n(x)$ is also given by the sum

$$P_n(x) = \sum_{m=0}^{M} \frac{(-1)^m (2n-2m)!}{2^n m!(n-m)!(n-2m)!}.x^{n-2m} \;\; , \;\; n = 0,\; 1,\; 2, ...$$

where $M = \frac{n}{2}$ or $\frac{n-1}{2}$, whichever is an integer.

(a) Prove that $xp_n'(x) = np_n(x) + p_{n-1}'(x)$.

(b) Express x^6 in terms of Legendre Polynomials.

(c) Show that (i) $p_n(-x) = (-1)^n p_n(x)$ and

(ii) $p_n'(-x) = (-1)^{n+1} p_n'(x)$.

(d) Show that

(i). $p_n'(1) = \frac{n(n+1)}{2}$

(ii). $p_n'(-1) = (-1)^{n+1} \frac{n(n+1)}{2}$.

(Hint: You may use the following recurrence relations, if necessary, without proof.)

$$P_n'(x) = xP_{n-1}'(x) + nP_{n-1}(x),$$
$$P_{n+1}'(x) - P_{n-1}'(x) = (2n+1)P_n(x),$$
$$(1-x^2)p_{n-1}' = n(xp_{n-1} - p_n),$$
$$(\mathrm{n}+1)p_{n+1} = (2n+1)xp_n - np_{n-1}$$
$$(x^2-1)(p_n') = n(xp_n - p_{n-1}).$$

Part B

1. The Laplace transform of a function $f(t)$, denoted by $L[f(t)]$ is defined as

$$L[f(t)] = F(s) = \int_0^\infty f(t)e^{-st}dt \quad \text{and} \quad L^{-1}\{F(s) = f(t)\}.$$

(a) Find the Laplace transform $L\{f(t)\}$ if $f(t) = e^{-t}\cos^2 t$.

(b) Find the inverse Laplace transform of $\frac{s-4}{4(s-3)^2+16}$.

(c) Using the convolution theorem, find the inverse Laplace transform of $\frac{s^2}{(s^2+a^2)(s^2+b^2)}$.

(d) Solve the following boundary value problem using the Laplace transform method:

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = x^2 + 5, \ y(0) = 1 \text{ and } y'(0) = 0.$$

2. (a) If $f(x)$ is a function defined in $-3 \le x \le 3$ write down the Fourier coefficients of $f(x)$.

(b) Express $f(x) = \frac{1}{2}(\pi - x)$ as Fourier series with period 2π to be valid in the interval

0 to 2π.

(c) Write the Fourier sine series of k in $(0, \pi)$.

(d) Let $f(x)$ be a function defined in the interval $0 < x < \pi$. The Fourier cosine series of $f(x)$ is

given by $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$,

where $a_0 = \frac{1}{\pi}\int_{x=0}^{\pi} f(x)dx$ and $a_n = \frac{2}{\pi}\int_{x=0}^{\pi} f(x)\cos nx dx$.

Show that $\frac{2}{\pi}\int_{x=0}^{\pi} [f(x)]^2 = 2a_0^2 + \sum_{n=1}^{\infty} a_n^2$.

3. (a) The Gamma function denoted by $\Gamma(p)$ corresponding to the parameter p is defined by the

improper integral $\Gamma(p) = \int_0^{\infty} e^{-t} t^{p-1} dt, \quad (p > 0)$.

(i) Evaluate $\frac{\Gamma(5).\Gamma(5.5)}{\Gamma(2.5)}$ and $\Gamma(-4.5)$.

(ii) Evaluate $\int_0^1 \frac{dx}{\sqrt{x \log \frac{1}{x}}}$ using Gamma function.

(b) The Beta function denoted by $\beta(p,q)$ is defined by $\beta(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1} dx$,

where p and q are positive parameters.

Use Gamma function and Beta function to evaluate the following integrals:

(i) $\int_0^{\frac{\pi}{2}} \sin^2\theta\cos^4\theta d\theta.$

(ii) $\int_0^2 \frac{x^2}{(2-x)^{\frac{1}{2}}}dx.$

4. Let $J_p(x)$ be the Bessel function of order p given by the expansion

$$J_p(x) = x^p \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+p} m!\Gamma(p+m+1)}.$$

(a) Prove that $\frac{d}{dx}\{x^{-p}J_p(x)\} = -x^{-p}J_{p+1}(x).$

(b) Express $J_{\frac{5}{2}}(x)$ in terms of sine and cosine functions.

$$\left(\text{Hint}: J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\sin x\right)$$

(c) Evaluate $\int x^4 J_1(x)dx$.

(Hint: You may use the following recurrence relations, if necessary, without proof.)

(i) $\frac{d}{dx}\{x^p J_p(x)\} = x^p J_{p-1}(x)$

(ii) $\frac{d}{dx}\{J_p(x)\} = J_{p-1}(x) - \frac{p}{x}J_p(x)$ or $xJ'_p(x) = xJ_{p-1}(x) - pJ_p(x)$

(iii) $J'_p(x) = \frac{p}{x}J_p(x) - J_{p+1}(x)$

(iv) $J'_p(x) = \frac{1}{2}\{J_{p-1}(x) - J_{p+1}(x)\}$

(v) $J_{p-1}(x) + J_{p+1}(x) = \frac{2p}{x}J_p(x)$