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THE OPEN UNIVERSITY OF SRI LANKA
BACHELOR OF SCIENCE DEGREE PROGRAMME – LEVEL 05
FINAL EXAMINATION - 2024/2025
PHU 5313 - ADVANCED ELECTROMAGNETISM
Duration: TWO (02) HOURS

Date 30th April 2025 **Time 9.30 am – 11.30 am**

$C = 2.99\times10^8\ ms^{-1}$, $\varepsilon_0 = 8.85\times 10^{-12}\ Fm^{-1}$, $\mu_0 = 4\pi\times 10^{-7}\ Hm^{-1}$

Answer any Four (04) questions only

1. In vector calculus, the Del operator is a vector differential operator used to define operations such as gradient, divergence, and curl. In Cartesian coordinates, it is defined as:

$$\vec{\nabla} = \frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}$$

(a) Define the following operations using the Del operator and state whether the resultant is a scalar or a vector:

i. Gradient of a scalar field

ii. Divergence of a vector field

iii. Curl of a vector field [5 marks]

(b) Given the scalar field $\phi(x,y,z) = xy^2 + z^2$, find the gradient $\vec{\nabla}\phi$. [4 marks]

(c) Let $\vec{F}(x,y,z) = x^2\hat{\imath} + yz\hat{\jmath} + z^2\hat{k}$. Find the divergence, $\vec{\nabla}\cdot\vec{F}$. [4 marks]

(d) For the vector field $\vec{A}(x,y,z) = yz\hat{\imath} + x\hat{\jmath} + y\hat{k}$, calculate the curl $\vec{\nabla}\times\vec{A}$. [6 marks]

(e) In electromagnetism, the magnetic field $\vec{B}$ is defined as the curl of a vector potential $\vec{B} = \vec{\nabla}\times\vec{A}$. Using the identity $\vec{\nabla}\cdot\left(\vec{\nabla}\times\vec{A}\right) = 0$, explain why the divergence of the magnetic field is zero.

What does this imply about the existence of magnetic monopoles? [6 marks]

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2. Let $\vec{A}(x,y,z) = (xy + yz + zx)$ and $\vec{B}(x,y,z) = (x + y + z)$ be vector fields, and let $\phi(x,y,z) = xyz$ be a scalar field.

(a) Verify the vector identity: $\nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B})$ [5 marks]

(b) Prove the identity: $\nabla \times (\phi\vec{A}) = \phi(\nabla \times \vec{A}) + (\nabla\phi) \times \vec{A}$ [5 marks]

(c) Using the identity: $\nabla \cdot (\phi\vec{A}) = \phi(\nabla \cdot \vec{A}) + \vec{A} \cdot (\nabla\phi)$, compute both sides explicitly and verify the result. [5 marks]

(d) Show that the curl of the gradient of any scalar field is zero: $\nabla \times (\nabla\phi) = 0$

Then verify this result for $\phi(x,y,z) = x^2y + yz^2$. [5 marks]

(e) In electromagnetism, Maxwell's equation $\nabla \cdot (\nabla \times \vec{B}) = 0$ holds for any magnetic field $\vec{B}$. Explain why this identity is always true using vector calculus. What fundamental physical principle does this represent in Maxwell's theory? [5 marks]

3. (a) State the Divergence Theorem and explain briefly what physical quantity it helps to calculate in a vector field. [4 marks]

(b) Let $\vec{F}(x,y,z) = (x + y + z)$. Use the Divergence Theorem to evaluate the outward flux of $\vec{F}$ across the surface of the sphere $x^2 + y^2 + z^2 = 1$. [6 marks]

(c) State Stokes' Theorem and describe how it relates to a vector field. [4 marks]

(d) Let $\vec{F}(x,y,z) = (-y + x, 0)$, and let S be the upper surface of the unit disk $x^2 + y^2 \leq 1$ in the xy-plane. Use Stokes' Theorem to evaluate the line integral $\oint_C \vec{F} \cdot d\vec{r}$, where C is the boundary of S oriented counter-clockwise. [6 marks]

(e) One of Maxwell's equations in differential form is: $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$. Using Stokes' Theorem, explain how this equation relates a changing magnetic field to the induced electric field in a loop. Briefly describe a physical situation where this occurs. [5 marks]

4. (a) A point charge +Q is located at the origin. Write down the expression for the electric field $\vec{E}(\vec{r})$ at a distance r from the origin in vacuum. Indicate the direction of the field. [4 marks]

(b) State the expression for the electric potential $V(\vec{r})$ at a point due to a continuous charge distribution $\rho(\vec{r}')$ in vacuum. [4 marks]

(c) Given the electric potential V(x, y, z), derive the expression for the electric field $\vec{E}(x, y, z)$ in terms of V. [4 marks]

(d) A charge q is moved from point A to point B in an electrostatic field. Show that the work done is independent of the path and express it in terms of the electric potential difference between A and B. [5 marks]

(e) Two different charge distributions, ρ_1 and ρ_2, are placed in the same volume V, giving rise to potentials V_1 and V_2, respectively. Use Green's Reciprocation Theorem to relate:

$$\int_V \rho_1(\vec{r})V_2(\vec{r})\, dv \quad \text{and} \quad \int_V \rho_2(\vec{r})V_1(\vec{r})\, dv$$

Then, use this theorem to compute the potential at the location of a grounded conducting sphere of radius R caused by a nearby point charge q placed at a distance d > R from the center. [8 marks]

5. (a) State Laplace's equation in three dimensions. Briefly explain the physical situations where Laplace's equation is applicable in electrostatics. [4 marks]

(b) State the uniqueness theorem for Laplace's equation. Why is it important in solving boundary value problems? [4 marks]

(c) Let V(x, y, z) satisfy Laplace's equation in a region R. Show that V has no local maxima or minima inside R, unless it is constant. [4 marks]

(d) A rectangular 2D region is defined for 0 < x < a and 0 < y < b. The potential on three sides of the rectangle is zero, and on the side y = b, the potential is given by. $V(x, b) = V_0 \sin\left(\frac{\pi x}{a}\right)$

Write down the boundary conditions for Laplace's equation in this region. [4 marks]

(e) Solve Laplace's equation for the region described in part (d) using the method of separation of variables. Find the potential V(x, y) in the rectangle. [9 marks]

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6.(a) Write down Maxwell's four equations in integral form and briefly explain the physical meaning of each. [6 marks]

(b) Define displacement current and explain its necessity.

Write down the modified Ampère's Law with the displacement current term. [4 marks]

(c) What does Gauss's Law for magnetism state? What does this imply about the existence of magnetic monopoles? [3 marks]

(d) Show how Maxwell's equations in vacuum lead to the wave equation for the electric field $\vec{E}$. [6 marks]

(e) An electromagnetic wave travels in vacuum along the positive z direction. Its electric field is given by, $\vec{E}(z,t) = E_0 \cos(kz - \omega t)\ \hat{x}$

Use Maxwell's equations to find the associated magnetic field $\vec{B}(z,t)$ Confirm that your result satisfies one of Maxwell's equations. [6 marks]

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